The Distributions of Eddy Viscosity and Turbulent Velocity in Pipe Flow

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Since Boussinesq (*1*) first introduced the concept of the eddy viscosity in 1877, many investigators have attempted to determine values of this quantity for the steady turbulent flow of incompressible fluids in pipes and between parallel flat plates. Corcoran et al. (*2*) have summarized some of the techniques that have been used to obtain these values, and have presented comparisons of measured and derived values.

Although no completely satisfactory expression exists to describe the turbulent time-smoothed velocity distribution in pipe flow, many investigators have employed analytical expressions for the velocity distribution in making eddy viscosity calculations. It is the purpose of this communication to summarize the conditions that the eddy viscosity concept and any valid turbulent velocity profile expression mutually impose on one another.

For pipe flow under the above mentioned restrictions, the eddy viscosity is defined by

$$Vg_c\,\tau_{rz} = -(v + \epsilon_T)\frac{dV_z}{dr} = -\epsilon_{TOT}\frac{dV_z}{dr} \qquad (1)$$

As Hinze (*3*), as well as other investigators, has shown, the axial component of the equation of motion may be integrated over the radius to produce the following shear stress distribution for turbulent pipe flow.

$$\tau_{rz} = \tau_o r/r_o \qquad (2)$$

In order to simplify the algebra involved, the dimensionless viscosity, radial position, velocity, and stress variables are employed. Substituting these variables, and (2) into (1),

$$\epsilon = -\chi\theta / \frac{du}{d\theta} \qquad (3)$$

Symmetry imposes the following conditions on $u(\theta)$

$$u(0) = 1 \quad \frac{du}{d\theta}(0) = 0 \qquad (4a)$$

while the assumption of no slip and conservation of mass add

$$u(1) = 0 \quad \int_0^1 u\theta d\theta = 2\langle u\rangle \qquad (4b)$$

Integration of (3) across the radius produces an often useful relationship between the eddy viscosity and the velocity.

$$\int_0^1 \epsilon du = \chi/2 \qquad (4c)$$

It is the purpose of this communication to examine what other higher ordered conditions are imposed on $\epsilon(\theta)$ by $u(\theta)$ and vice versa.

WALL EFFECTS

At the wall, molecular effects predominate, ($\epsilon = 1$) and

$$\frac{du}{d\theta}(1) = -\chi \qquad (5)$$

Further

$$\frac{d\epsilon}{d\theta} = -\chi\left[\frac{\frac{du}{d\theta} - \theta\frac{d^2u}{d\theta^2}}{\left(\frac{du}{d\theta}\right)^2}\right] \qquad (6)$$

At the wall

$$\frac{d\epsilon}{d\theta} = \chi\left[\frac{\chi + \frac{d^2u}{d\theta^2}(1)}{\chi^2}\right] \qquad (7)$$

This relationship may be manipulated to produce the inverse relations for the wall derivatives.

$$\frac{d\epsilon}{d\theta}(1) = 1 + \frac{\frac{d^2u}{d\theta^2}(1)}{\chi} \qquad (8a)$$

$$\frac{d^2u}{d\theta^2}(1) = \chi\left[\frac{d\epsilon}{d\theta}(1) - 1\right] \qquad (8b)$$

CENTER

At the pipe center

$$\epsilon = \lim_{\theta\to 0} -\chi\frac{\theta}{du/d\theta} \qquad (9)$$

Applying L'hospital's rule, the inverse center relations are produced.

$$\epsilon(0) = \frac{-\chi}{\frac{d^2u}{d\theta^2}(0)} \qquad (10a)$$

An alternate expression that is less awkward may also be written.

$$\epsilon(0) = -\frac{\chi}{2\frac{du}{d\theta^2}(0)} \qquad (10b)$$

Graphical differentiation of velocity distribution data will show that $\frac{d^2u}{d\theta^2}(0)$ is finite and negative, causing $\epsilon(0)$ to be nonzero and positive.

Further, for continuity and symmetry it is necessary that $d\epsilon/d\theta$ and all other odd order derivatives of ϵ be zero at the tube center.

Differentiating (3), and applying L'hospital's rule three times, the following is obtained at $\theta = 0$.

$$\frac{d\epsilon}{d\theta}(0) = -\frac{d^3u}{d\theta^3}(0)\left[\frac{\epsilon(0)^2}{2\chi}\right] \qquad (11)$$

This shows that the condition for symmetry and continuity on ϵ imposes the condition on $u(\theta)$

$$\frac{d^3u}{d\theta^3} = 0 \quad \theta = 0 \qquad (12)$$

It should be pointed out that it is not necessarily obvious on physical grounds that ϵ should have continuous derivatives across the tube center. These relations show only what is required of the distribution expressions if continuity is to be indeed satisfied.

TABLE 1
Turbulent velocity
$u = V_s/V_{s\,max}$

$\int_0^1 u\theta d\theta = 2\langle u\rangle$		
$u(0) = 1$	$u(1) = 0$	
$\frac{du}{d\theta}(0) = 0$	$\frac{du}{d\theta}(1) = -\chi$	$\frac{du}{d\theta}(\theta_c) = \theta_c \frac{d^2u}{d\theta^2}(u_c)$
$\frac{d^2u}{d\theta^2}(0) = \frac{-\chi}{\epsilon(0)}$	$\frac{d^2u}{d\theta^2}(1) = \chi\left[\frac{d\epsilon}{d\theta}(1) - 1\right]$	
$\frac{d^3u}{d\theta^3}(0) = 0$		$\frac{d^3u}{d\theta^3}(\theta_c) = \frac{\frac{d^2\epsilon}{d\theta^2}(\theta_c)\frac{du}{d\theta}(\theta_c)^2}{\chi\theta_c}$
$\frac{d^4u}{d\theta^4}(0) = \frac{\epsilon''(0)\chi}{\epsilon(0)^2}$		

Eddy viscosity

$$\epsilon = \frac{\epsilon_{TOT}}{\nu}$$

$\epsilon(0) = \frac{-\chi}{\frac{d^2u}{d\theta^2}(0)}$	$\epsilon = \frac{-\chi\theta}{\frac{du}{d\theta}}$	$\int_0^1 \epsilon du = \frac{\chi}{2}$
$\epsilon(0) = \frac{-\chi}{2\frac{du}{d\theta^2}(0)}$	$\frac{d\epsilon}{d\theta}(1) = 1 + \frac{\frac{d^2u}{d\theta^2}(1)}{\chi}$	$\frac{d\epsilon}{d\theta}(\theta_c) = 0$
$\frac{d\epsilon}{d\theta}(0) = 0$	$\frac{d^2\epsilon}{d\theta^2}(\theta_c) = \frac{\chi\theta_c \frac{d^3u}{d\theta^3}(\theta_c)}{\left[\frac{du}{d\theta}(\theta_c)\right]^2}$	
$\frac{d^2\epsilon}{d\theta^2}(0) = \frac{\frac{d^4u}{d\theta^4}(0)\chi}{\left[\frac{d^2u}{d\theta^2}(0)\right]^2}$		

Experimental evidence from many investigators shows that ϵ decreases near the center line. This, of course, implies that if $\frac{d\epsilon}{d\theta}(0) = 0$, then $\frac{d^2\epsilon}{d\theta^2}$ is necessarily positive at the center. Differentiating (6), applying L'hospital's rule, breaking up the limit into two parts, employing (11), the following is obtained.

$$\frac{d^2\epsilon}{d\theta^2}(0) = \chi \frac{\frac{d^4u}{d\theta^4}(0)}{\left[\frac{d^2u}{d\theta^2}(0)\right]^2} \quad (13)$$

Substituting from (10*b*),

$$\frac{d^2\epsilon}{d\theta^2}(0) = \frac{d^4u}{d\theta^4}(0)\frac{\epsilon(0)^2}{\chi} \quad (14)$$

This implies that $\frac{d^4u}{d\theta^4}(0)$ must be positive if ϵ reaches a minimum at $\theta = 0$.

$0 < \theta < 1$

If ϵ reaches a minimum at the center, then, as has been shown by many investigators, it reaches a maximum at a value of θ away from the center, denoted by θ_c. From (6), an expression at θ_c may then be written.

$$\frac{du}{d\theta}(\theta_c) = \theta_c \frac{d^2u}{d\theta^2}(\theta_c) \quad (15)$$

At θ_c, ϵ reaches a maximum, $\frac{d\epsilon}{d\theta} = 0$, and $\frac{d^2\epsilon}{d\theta^2}$ must be negative. Differentiating, taking the limit, and employing (15), the following is obtained.

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Chemical Reactor Stability by Liapunov's Direct Method

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In two recent articles (*1, 2*), Berger and Perlmutter discuss the use of Liapunov functions to determine the stability range of a chemical reactor. Two points, however, require some additional comment. First, the conditions presented to determine the region of asymptotic stability are overly conservative. Second, a less restrictive definition of Krasovskii's theorem for the generation of Liapunov functions allows a search for a broader stable region.

CONDITIONS FOR STABILITY

In Liapunov's direct method, asymptotic stability of a system is assured in a region of the phase space enclosed by an analytic positive definite function, V, provided that its time derivative, $\dot{V}$, is negative throughout this region (except for the trivial solution at the focus). After constructing a quadratic function of the time derivatives of the state variables, the authors (*1*) use Sylvester's conditions to determine the sign of the derivative of the Liapunov function, $\dot{V}$. However, the Sylvester conditions are necessary only for positive definiteness throughout the phase space; they do not delineate the region in which the quadratic form is positive. For example, the Sylvester conditions are not fulfilled at any point for the function $F = x^2 + 3\,xy + y^2$; yet F is positive throughout a considerable part of the xy plane. Thus, the derivative of the Liapunov function as defined in reference *1* is

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$$\frac{d^2\epsilon}{d\theta^2}(\theta_c) = \chi \frac{\theta_c \dfrac{d^3u}{d\theta^3}(\theta_c)}{\left[\dfrac{du}{d\theta}(\theta_c)\right]^2} \quad (16)$$

This implies that $\frac{d^3u}{d\theta^3}(\theta_c)$ must be negative.

Since $\frac{du}{d\theta}$ is negative or zero everywhere, as is $\frac{d^2u}{d\theta^2}$, the following may be written.

$$\theta_c < \theta < 1 \quad \frac{d^2u}{d\theta^2}\theta < \frac{du}{d\theta} \quad (17)$$

$$0 < \theta < \theta_c \quad \frac{du}{d\theta} < \theta \frac{d^2u}{d\theta^2} \quad (18)$$

$$\theta = 1 \quad \frac{d^2u}{d\theta^2}(1) < -\chi \quad (19)$$

A total of thirteen constraining relationships on $u(\theta)$ with an associated eight inverse relations on $\epsilon(\theta)$ has been presented. Obviously no existing expression proposed for the turbulent velocity distribution in pipes meets these numerous conditions. It is believed, however, that it is constructive to have in hand what the mutually constraining conditions are on these two distributions. The relationships that have been discussed are summarized in Table 1.

NOTATION

r = radial distance, L
r_w = pipe radius, L
u = dimensionless velocity, V_z/V_{zmax}
V = specific volume, M/L^3
V_z = time smoothed axial velocity, L/T
ϵ = dimensionless viscosity, ϵ_{TOT}/v
ϵ_T = eddy kinematic viscosity, L^2/T
ϵ_{TOT} = total viscosity, L^2/T
v = molecular kinematic viscosity, L^2/T
θ = dimensionless radial position, r/r_w
τ_{rz} = shear stress, F/L^2
τ_w = wall shear stress, F/L^2
χ = dimensionless parameter, $\dfrac{r_o \tau_w V g_c}{V_{zmax} v}$

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